A Model for the Determination of the Natural Vibration Frequencies of Self-Supporting Lattice Towers

Prof. N. N. Osadebe, Dr. M. E. Onyia and M. C. Nwosu
Department of Civil Engineering
University of Nigeria, Nsukka. Nigeria.

ABSTRACT

Self-supporting lattice towers are used mainly for supporting telecommunication antennae, wind turbines and high-voltage electrical transmission infrastructure. The four-legged self-supporting towers are widely used worldwide mainly for telecommunication purposes to increase coverage and network consistency. The failure of such structure could lead to loss of lives and property, as well as disruption of services. The dynamic analysis of self-supporting lattice towers therefore demands a high degree of reliability. In view of their significant role, it is necessary to evolve alternative methods of dynamic analysis of self-supporting lattice towers which will give acceptable results.

This paper proposes a model for the determination of the natural vibration frequencies of self-supporting lattice towers subjected to the dynamic action of wind loads. The proposed model idealizes the lattice tower as an equivalent solid beam-column whose cross-sectional dimensions are the unknowns to be determined. The expression $f_o = \frac{\pi b^2}{10^2}\sqrt{\frac{E}{3m}}$ is proposed by the model for the computation of the fundamental natural vibration frequency of self-supporting lattice towers whose equivalent beam-column structure has a dimension of b at its free end.

Keywords: Beam-column, Lattice tower, Natural frequency, Resonance, Truss

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1.0 INTRODUCTION

Towers are tall steel frame structures used for different purposes such as installation of equipment for telecommunication, radio transmission, satellite reception, air traffic control, television transmission, power transmission, flood lights, meteorological measurements, etc.

Lattice towers act as vertical trusses and resist wind by cantilever action. The bracing members, which are arranged in many forms, are designed to resist tensile or compressive forces. The height of a tower is normally several times larger than the horizontal dimensions. Towers are therefore more likely to fail by bending due to the horizontal action of wind. They act as cantilever trusses since they are usually clamped at the base, and are designed to carry wind and seismic loads. The vertical load is as a result of self-weight and the equipment installed on the tower.

This paper proposes a model for the determination of the natural vibration frequencies of self-supporting lattice towers by replacing the actual tower with an equivalent beam-column. In natural frequency calculations, the designer is interested in all those frequencies which are likely to coincide with the frequencies of the applied forces (such as that due to wind, seismic forces, vibrating machinery, etc.) which have sufficient energy to excite one or more of the natural frequencies of the structure. The main concept used in determining the natural frequency is that if an applied vibratory force with a frequency equal to the natural frequency of the structure acts on the structure, then the structure resonates. In the absence of damping, when the structure is resonating, the displacements tend to infinity.

2.0 STRUCTURAL MODELLING

The structural model is a solid beam-column of exactly the same height and lateral deflection curve as the actual self-supporting lattice tower. The cross sections

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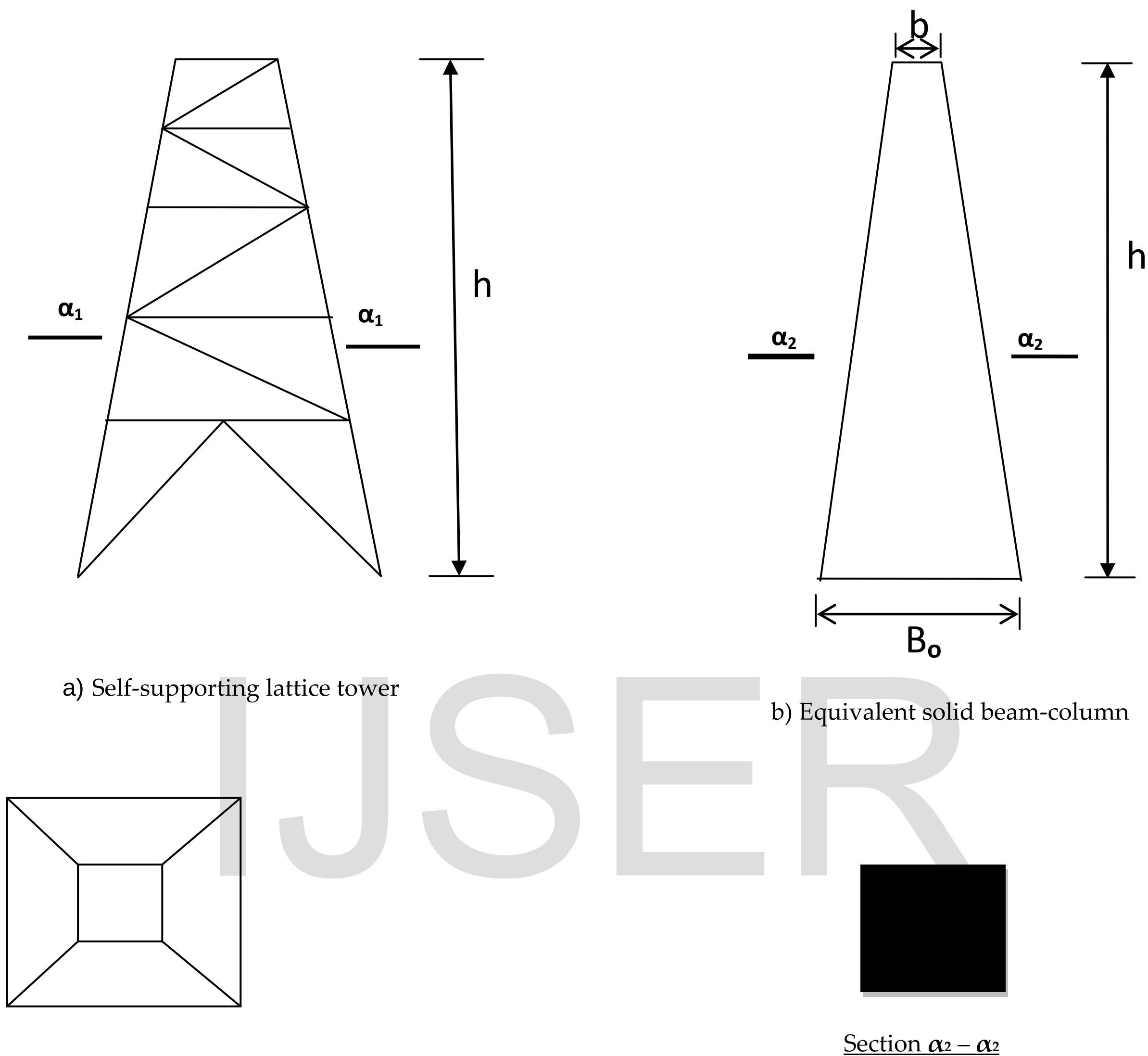
of both the self-supporting tower and the equivalent structure should be similar but must not be equal in dimensions. (Fig. 1)



a) Self-supporting lattice tower

b) Equivalent solid beam-column

Section $\alpha_1 - \alpha_1$

Section $\alpha_2 - \alpha_2$

Fig. 1 – Structural modelling of lattice tower

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Since the self-supporting truss tower is normally prevented from movement at its base, the equivalent solid beam-column is analysed as a linearly-varying cantilever beam. The equivalent beam-column is assumed to have the same values of lateral deflection (sway) under the action of the same applied loads at exactly the same points along its length as the self-supporting lattice tower. The analysis thus considers the failure of the tower structure as a whole, rather than the failure of the individual truss members.

3.0 MATHEMATICAL MODELLING

The analysis consists of the following steps

(i) Select an appropriate structural model (equivalent solid beam-column) that best suits the actual structure (self-supporting lattice tower) under consideration. The model must have exactly the same height and cross-sectional shape as the actual structure.

(ii) Perform an analysis of the actual self-supporting lattice tower with the given dimensions and loadings to determine the

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numerical values of lateral deflection (sway) along its length.

(iii) Using the determined deflection valves at known points on the actual structure, determine the unknown cross-sectional dimensions of the equivalent solid beam-column by equating deflections at the same points along the length of the equivalent structure. Thus, the self-supporting lattice tower and the equivalent solid beam-column have to be analyzed under the action of the same loadings acting at the same points and direction.

(iv) Perform a dynamic analysis of the equivalent solid beam-column to determine its natural vibration frequencies.

3.1 Cross-Sectional Properties Of The Equivalent Solid Beam-Column

Consider a solid beam-column structure of height h with a linearly-tapering cross-section and fixed at its base, (Fig. 2)

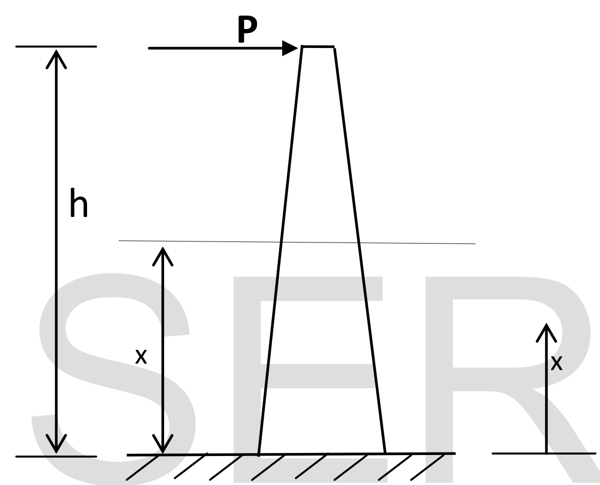

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Fig.2- Solid beam-column with linearly-tapering cross-section

A horizontal force P is applied at its free end. The bending moment along the cantilever solid beam is

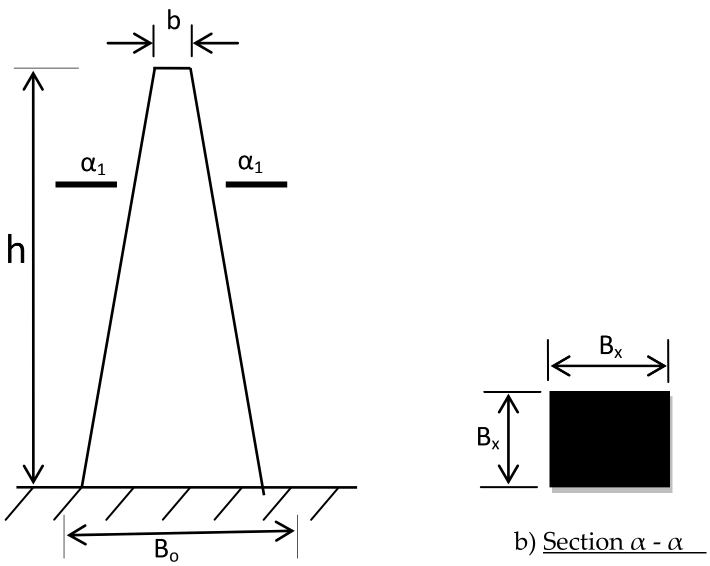
$$M_x = P\,(h - x) \qquad (1)$$

From theory of structures, strain energy due to the applied load is

$$U_B = \int \frac{M_x^2\,dx}{2EI_x} \qquad (2)$$

From Castigliano's theorem, the deflection of the member is expressed as

$$\delta_B = \frac{\partial U_B}{\partial P} = \int \frac{M_x}{EI_x}\frac{\partial M_x}{\partial P}\,dx \qquad (3)$$



a) Solid beam-column

b) Section α - α

Fig.3 –Cross-sectional dimensions of the equivalent beam-column

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The linearly-tapering dimension of the beam-column can be expressed as

$$B_x = a + cx \tag{4}$$

where B_X is the width of the cross-section at any point x along the length of the equivalent beam-column structure. To determine the values of the constants a and c in Eq.(4), we need to consider the boundary conditions of the equivalent beam-column structure under consideration, (Fig.3):

(i) At x = 0 (base), $B_x = B_o$

(ii) At x = h (top), $B_x = b$

$$\therefore \; B_X = \frac{B_0 h + (b - B_0)x}{h} \tag{5}$$

The moment of inertia for the equivalent solid beam can be expressed in terms of B_X.

Moment of Inertia, $I_x = \frac{B_X (B_x)^3}{12}$

$$\text{i.e. } I_x = \frac{[B_o h + (b - B_o)x]^4}{12h^4} \tag{6}$$

Putting $B_o h = \alpha_o$ and $b - B_o = \beta$

Then the expressions for B_x and I_x can be expressed as follows:

$$B_x = \frac{\alpha_0 + \beta x}{h} \tag{7}$$

$$I_x = \frac{(\alpha_0 + \beta x)^4}{12h^4} \tag{8}$$

The strain energy of the equivalent structure is given by Eq.(2):

$$\text{i.e } U_B = \int \frac{M_x^2 \, dx}{2EI_x}$$

Substituting for M_x and I_x in the strain energy equation:

$$U_B(x) = \frac{6P^2h^4}{E} \int \frac{(h-x)^2}{(\alpha_0 + \beta x)^4} dx \tag{9}$$

From Castigliano's theorem, the deflection of the equivalent structure is given by Eq.(3):

$$Y_E(x) = \frac{\partial U_B(x)}{\partial P} = \int \frac{M_x}{EI_x} \left(\frac{\partial M_x}{\partial P}\right) dx$$

But $M_x = P(h - x)$ and $I_x = \frac{(\alpha_0 + \beta x)^4}{12h^4}$

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Thus, $\frac{\partial M_x}{\partial P}$ = h-x

$\therefore \; Y_E(x) = \int \frac{M_x}{EI_x}\left(\frac{\partial M_x}{\partial P}\right)dx$

i.e. $Y_E = \frac{12h^4P}{E}\int \frac{(h-x)^2}{(\alpha_0+\beta x)^4}dx$ (10)

Integrating by partial fractions,

$$Y_E(x) = \left(\frac{-12h^4P}{E\beta^3}\right).\left[\frac{1}{(\alpha_0+\beta x)} - \frac{(\alpha_0+\beta h)}{(\alpha_0+\beta x)^2} + \frac{(\alpha_0+\beta h)^2}{3(\alpha_0+\beta x)^3} - \frac{1}{\alpha_0} + \frac{(\alpha_0+\beta h)}{\alpha_0^2} - \frac{(\alpha_0+\beta h)^2}{3\alpha_0^3}\right] \quad (11)$$

where

$\alpha_o = B_o h$

$\beta = b - B_o$

The proposed model (i.e the beam-column) can only be said to be equivalent to the actual self-supporting lattice tower if its deflection curve under the action of the same loading is the same as that of the actual tower. Therefore, the self supporting lattice tower should be analysed statically for deflection along its length and the values at x = h and x = $\frac{h}{2}$ equated to the above expression for deflection of the equivalent solid beam-column (i.e Eq.12) to determine the unknown values of its cross-section, b and B_o.

If the deflection of the free end (tip) is Y, then putting x = h in Eq.(11), and noting that

$\alpha_o = B_o h$ and $\beta = b - B_o$, we get

$Y_1 = \frac{4h^3P}{EbB_0^3}$

$b = \frac{C}{Y_1 B_0^3}$ (12)

where $c = \frac{4h^3P}{E}$ (13)

If the deflection at the middle is $Y_{1/2}$, then putting x = $\frac{h}{2}$ in Eq.(11), we have that:

$$\alpha_1 B_0^{12} + \alpha_2 B_0^8 + \alpha_3 B_0^4 + \alpha_4 = 0 \quad (14)$$

where,

$$\left.\begin{aligned} \alpha_1 &= Y_1^3 Y_{1/2} \\ \alpha_2 &= CY_1^2 (Y_{1/2} - 3Y_1) \\ \alpha_3 &= 3C^2Y_1 (Y_{1/2} - Y_1) \\ \alpha_4 &= C (Y_{1/2} - C^2Y_1) \end{aligned}\right\} \quad (15)$$

The values of b and B_0 are then determined from Eqns.(12) and (14).

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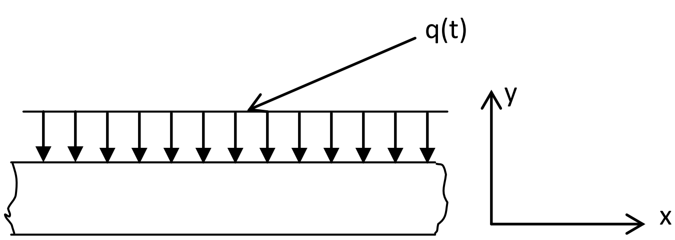
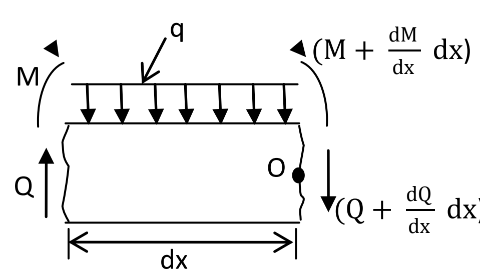
3.2 Vibration Frequencies Of The Equivalent Beam

The natural vibration frequencies of the self-supporting lattice tower can be determined by considering the flexural vibration of the equivalent solid beam (Fig.4):



a) Beam under flexural



b) Elemental beam

Fig.4 – Flexural Vibration of Beam.

Consider an infinitesimal length dx of the beam (Fig. 4b):

From vertical equilibrium:

$$\left(Q + \frac{dQ}{dx}dx\right) - Q - qdx = 0$$

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i.e $\frac{dQ}{dx} - q = 0$ (i)

Taking moments about point O:

$$(M + (qdx).\frac{dx}{2} + Qdx - \left(M + \frac{dM}{dx}dx\right) = 0$$

i.e $q\frac{dx^2}{2} + Qdx - \frac{dM}{dx}dx = 0$

Ignoring terms of second order, we have:

$$\frac{dM}{dx} - Q = 0 \qquad \text{(ii)}$$

Differentiating Eq.(ii) with respect to x:

$$\frac{d^2M}{dx^2} - \frac{dQ}{dx} = 0$$

But from Eqn.(i), $\frac{dQ}{dx} = q$

$$\therefore \frac{d^2M}{dx^2} = q \qquad (16)$$

Eq.(16) is the differential equation governing the bending of beams.

From D'Alembert's principle,

q = (externally applied load + inertia force) per unit length.

If a beam with mass per unit length of m is vibrating, then the inertia force per unit length is given by

F = -ma

where a is the acceleration

i.e. $F = -\frac{m\,\partial^2 y}{\partial t^2}$ (iii)

Thus, q = (externally applied load - $\frac{m\partial^2 y}{\partial t^2}$)

where y is the lateral displacement and $\frac{\partial^2 y}{\partial t^2}$ is the acceleration in the lateral direction.

Note that partial derivative is used here because the displacement y is a function not only of x but also of time t.

The bending moment M is given by

$$M = EI\frac{\partial^2 y}{\partial x^2} \qquad \text{(iv)}$$

Substituting for M and q in the different equation of motion, Eq.(16):

$$\frac{\partial^2}{\partial x^2}\left[\frac{EI\,\partial^2 y}{\partial x^2}\right] = \left[q(t) - \frac{m\,\partial^2 y}{\partial t^2}\right] \qquad \text{(v)}$$

where q(t) is the time-varying external load per unit length.

Considering free vibration, q(t) = 0, Then the differential equation for free vibration becomes

$$\frac{EI\,\partial^4 y}{\partial x^4} + \frac{m\,\partial^2 y}{\partial t^2} = 0 \qquad \text{(vi)}$$

If the system is undamped, then the motion is simple harmonic. Therefore,

$y(x,t) = Y(x)\cos\omega t$

where Y(x) is the amplitude of motion and ω is the circular frequency of vibration.

Substituting for y in Eq.(iv) and simplifying, the differential equation of motion becomes

$$\frac{EId^4y}{dx^4} - m\omega^2 Y(x) = 0 \qquad (17)$$

where m is the distributed mass per unit length.

Eq.(17) is the basic differential equation governing the undamped free vibration of beams.

The solution to the above dynamic flexural differential equation is given by

$$y = C_1 \sin\lambda x + C_2 \cos\lambda x + C_3 \sinh\lambda x + C_4 \cosh\lambda x \qquad (18)$$

where C_1 to C_4 are constants of integration to be determined from the boundary conditions of the beam.

For the cantilever beam,

(i) y(x) = 0 at x = 0
θ(x) = 0 at x = 0, and

(ii) M(x) = 0 at x = h
Q(x) = 0 at x = h

where $\theta(x) = \frac{dy}{dx}$, $M(x) = -\frac{EIdy}{dx^2}$ and $Q(x) = -\frac{EId^3y}{dx^3}$

λ in Eqn. (18) is given by

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$$\lambda = \sqrt[4]{\frac{m\omega^2}{EI}} \quad (19)$$

or $\omega = \lambda^2\sqrt{\frac{EI}{m}}$

$$\text{or} \quad f = \frac{\omega}{2\pi} = \frac{\lambda^2}{2\pi}\sqrt{\frac{EI}{m}} \quad (20)$$

For a cantilever beam, the natural frequencies occur at $\lambda = \frac{n\pi}{5}$, n = 1, 2, 3, 4

Thus, $f = \left(\frac{n\pi}{5}\right)^2 . \frac{1}{2\pi}\sqrt{\frac{EI}{m}}$

For a linearly-tapering beam, $I = I_x = \frac{(\alpha_0 + \beta x)^4}{12h^4}$

$$f = \left(\frac{n\pi}{5}\right)^2 . \frac{1}{2\pi}\,\frac{(\alpha_0+\beta x)^2}{2h^2}\sqrt{\frac{E}{3m}}$$

At the free end of the cantilever, x = h. Noting that $\alpha_o = B_o h$ and $\beta = b - B_o$, then

$$f = \frac{n^2\pi b^2}{100}\sqrt{\frac{E}{3m}}$$

n = 1 for fundamental natural frequency. Thus,

$$f_o = \frac{\pi}{100}b^2\sqrt{\frac{E}{3m}} \quad (21)$$

where m is the mass per unit length and b is the cross-sectional dimension of the equivalent beam at its free end.

4.0 PRESENTATION OF RESULTS

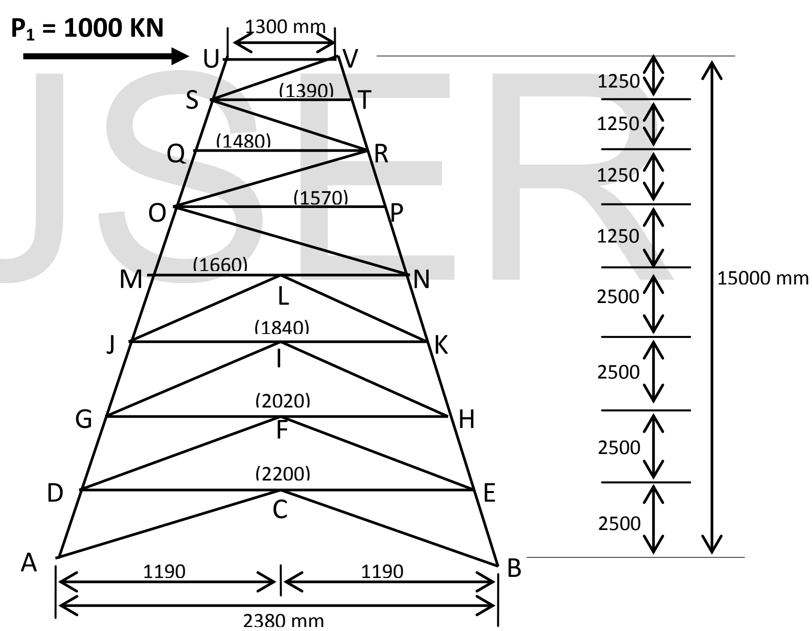

Fig.5 – Self-supporting lattice tower under load

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The self-supporting lattice tower in Fig. 5 is subjected to a horizontal load P_1 = 1000KN at its free end (Point U).

The tower has the following properties:

(i) 128 truss members with total volume = 0.9684m^3

(ii) Density = 7850 kg/m^3

(iii) Young's Modulus, E = 210 x 10^6 KN/m^2

Table 1 gives the results of the deflection analysis of the self-supporting lattice tower.

Table 1 – Deflection Values for Self-supporting Lattice Tower

S/N	Joint	Height from base (m)	Deflection (m)
1	U	15.00	0.59133378
2	S	13.75	0.390101219
3	Q	12.50	0.292947928
4	O	11.25	0.19851868
5	M	10.00	0.146828164
6	J	7.50	0.073638858
7	G	5.00	0.029030528
8	D	2.50	0.004289548
9	A	0.00	0.00

From Table 1,

(i) Deflection at the free end (tip), $(x = h) = Y_1 = 0.59133378$m

(ii) Deflection at mid-height, $(x = \frac{h}{2}) = Y_{1/2} = 0.073638858$m

From Eq. (13), $c = \frac{4h^3P}{E} = \frac{4(15^3)(1000)}{210x10^6} = 0.0643\ m^5$

From Eqs. (15),

$\alpha_1 = 0.0152, \alpha_2 = -0.0382, \alpha_3 = -0.003797,$

$\alpha_4 = 0.004578$

Therefore Eq. (14) becomes:

$0.0152\ B_o^{12} - 0.0382B_o^8 - 0.003797B_o^4 + 0.004578 = 0$

Using Newton-Raphson method,

$B_o = 0.750852$ m

Substituting in Eq. (12),

$b = \frac{C}{Y_1 B_0{}^3} = 0.2569$ m

a) <u>Natural Frequency of Equivalent Beam</u>

The fundamental natural vibration frequency of the equivalent beam is given by Eq.(21):

$$f_o = \frac{\pi b^2}{100}\sqrt{\frac{E}{3m}}$$

where m is the mass per unit length.

Considering unit length of the equivalent beam (frustum),

Then mass per unit length is

$$m = \varrho[\frac{h}{3}\ (b + B_o + \sqrt{(bB_o)}\ ]$$

$$= 7850\ [\frac{15}{3}\ (0.2569 + 0.750852 + \sqrt{(0.2569 \times 0.750852)}$$

$$= 56{,}792.743\ \text{kg/m}$$

Thus, fundamental natural vibration frequency is

$$f_o = \frac{\pi b^2}{100}\sqrt{\frac{E}{3m}}$$

$$= \frac{\pi(0.2569)^2}{100}\sqrt{\frac{210x10^9}{3(56782.743)}}$$

i.e. $f_o = 2.30$ Hz

b) <u>Natural frequency of the Lattice Tower</u>

From Table 1,

The deflection of the free end, $\Delta_u = 0.59133378$m

Force applied at the free end, $F_u = 1000$ KN

Hence, stiffness of the structure, $K = \frac{F_u}{\Delta_u} = \frac{1000x10^3}{0.59133378}$

i.e. K = 1691092.296 N/m

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The mass of the truss tower in Fig. 5 is computed from the given information on the cross sectional areas and lengths of the 128 truss members

Mass, $M = \varrho[\sum_1^n A_i L_i]$

where n = number of truss members

ϱ = density = 7850 kg/m^3

M = 7850 x (0.9684m^3) = 7602 kg

For the linearly-tapering cantilever tower structure, the fundamental natural frequency is given by: (Bhatt et. al, 1994)

$$f = \frac{c}{2\pi}\sqrt{\frac{K}{M}}\text{ , where c = 0.975}$$

$$= \frac{0.975}{2\pi}\sqrt{\frac{1691092.296}{7602}}$$

$$\text{i.e.}\quad f = 2.31\text{ Hz}$$

The summary of the results is presented in Table 2 below.

Table 2 – Comparison of Results

Description	**Lattice Tower**	**Equivalent beam-column**	**% Difference**
Top dimension	1300 mm	256.90 mm	
Base dimension	2380 mm	750.852 mm	
Mass	7602 kg	56792.74 kg/m	
Natural Frequency	2.31	2.30	0.43

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5.0 CONCLUSIONS

A comparison of the fundamental natural vibration frequency values of the actual lattice tower and the proposed model shows a marginal percentage difference of 0.43%.The proposed model is acceptable since it gives a lower-bound value of the vibration frequency, which is a welcome safeguard against resonance. The derived model expressions can also be easily modified to analyze towers of different cross-sectional shapes, such as circular and triangular-shaped towers.

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